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# Non-steady membrane creep of circular plates

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In a previous paper [1] plastic deformation of a plane circular membrane was studied, assuming a non-linear relationship between stress and strain rate tensors and finite deformation. The treatment was formally within the Eulerian conception of flow of matter and the problem considered as stationary, although the phenomenon as such is essentially a non-stationary one. Strictly speaking, the problem solved in [1] is one of non-linear elasticity rather than one of plasticity or creep if only $u$, $v$, $w$, be interpreted as displacements instead of velocities and $\varepsilon_r$, $\varepsilon_t$ as strains instead of strain rates.[1]

In this paper we shall show how to modify the mathematical treatment so as to cover the physical problem of incipient creep of a metallic membrane. The equations of [1] will be quoted as $(1)_{\mathrm{I}}$, $(2)_{\mathrm{I}}$, ...

Instead of the equations $(1)_{\mathrm{I}}$, $(2)_{\mathrm{I}}$, $(5)_{\mathrm{I}}$ in [1] we shall now assume (fluxions, denoting derivatives with respect to the time $t$)

$$\left.\begin{aligned} \dot{\varepsilon}_r &= \frac{d\dot{u}}{dr} + \frac{dw}{dr}\cdot\frac{d\dot{w}}{dr} \\ \dot{\varepsilon}_t &= \frac{\dot{u}}{r} \end{aligned}\right\} \qquad (1)$$

$$\left.\begin{aligned} \dot{\varepsilon}_r &= F(\sigma_e)(\sigma_r - \sigma_t/2) \\ \dot{\varepsilon}_t &= F(\sigma_e)(\sigma_t - \sigma_r/2) \end{aligned}\right\} \qquad (2)$$

$$\dot{\varepsilon}_e^2 = 4\left[\dot{\varepsilon}_r^2 + \dot{\varepsilon}_t^2 + \dot{\varepsilon}_r\dot{\varepsilon}_t\right]/3 \qquad (3)$$

whereas equations $(3)_{\mathrm{I}}$, $(4)_{\mathrm{I}}$, $(6)_{\mathrm{I}} \ldots (8)_{\mathrm{I}}$ remain unchanged, supposing nowmore $u$, $v$, $w$ to be the displacement vector and $\varepsilon_r$, $\varepsilon_t$ to be the strains in radial and tangential direction, so that $\dot{\varepsilon}_r$, $\dot{\varepsilon}_t$ are the strain rates.

If we introduce again dimensionless variables according to $(9)_{\mathrm{I}}$, we arrive at the modified system

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[1] The title of [1] is justified from considerations of primary creep, cf. [2].
This opportunity may be utilized to correct two misprints in [1], viz. Equation (17) reads

$$[R^2 - RT + T^2]^{\frac{n-1}{n}} \ldots; \text{ should read } [R^2 - RT + T^2]^{\frac{n-1}{2}} \ldots$$

Equation (18): The first square bracket reads

$$[3n+6)R_2 + \cdots]; \text{ should read } [(3n+6)R_2 + \cdots]$$

$$\frac{d}{d\varrho}(\varrho R) = T \tag{4}$$

$$R\frac{dW}{d\varrho} = -\frac{\varrho}{2} \tag{5}$$

$$\frac{d\dot U}{d\varrho} + \frac{dW}{d\varrho}\frac{d\dot W}{d\varrho} = (R^2 - RT + T^2)^{\frac{n-1}{2}}(R - T/2) \tag{6}$$

$$\frac{\dot U}{\varrho} = (R^2 - RT + T^2)^{\frac{n-1}{2}}(T - R/2) \tag{7}$$

with the boundary condition (as before $a$ = plate radius, $h$ = plate thickness and $p$ = lateral pressure) $U = 0$, $W = 0$ for $\varrho = \varrho_1 = ap/\sigma_c h$ and initial conditions to be specified later. If we assume a solution similar to $(15)_{\text{I}}$, $(16)_{\text{I}}$

$$\begin{aligned} R &= S[1 + R_1\varrho^2 + R_2\varrho^4 + \cdots] \\ T &= S[1 + 3R_1\varrho^2 + 5R_2\varrho^4 + \cdots] \end{aligned} \tag{8}$$

we have already satisfied (4) but shall have to assume from the outset $S$, $R_1$, $R_2$ ... as functions of $t$ to be determined. From (7) we obtain

$$\begin{aligned} \dot U = \frac{S^n}{2}\{\varrho + (2n+3)R_1\varrho^3 + [(3n+6)R_2 + (4n^2 + 11n - 15)R_1^2/2]\varrho^5 + \\ + \cdots (R)_m \varrho^{2m+1} \cdots\} \end{aligned} \tag{9}$$

and from (5), (8) and (9)

$$\begin{aligned} \frac{d\dot U}{d\varrho} + \frac{dW}{d\varrho}\frac{d\dot W}{d\varrho} = \frac{1}{4S}\left\{\left(\frac{1}{S}\right)^{\cdot}\varrho^2 - \left[2R_1\left(\frac{1}{S}\right)^{\cdot} + \left(\frac{R_1}{S}\right)^{\cdot}\right]\varrho^4 - \cdots\right. \\ \left. - \left[(R)_{m-1}\left(\frac{1}{S}\right)^{\cdot} + (R)^{\cdot}_{m-1}\frac{1}{S}\right]\varrho^{2m} - \cdots\right\} + \frac{S^n}{2}\{1 + 3(2n+3)R_1\varrho^2 + \\ + 5[(3n+6)R_2 + (4n^2 + 11n - 15)R_1^2/2]\varrho^4 + \cdots + (R')_m\varrho^{2m} + \cdots\} \end{aligned} \tag{10}$$

Here $(R)_m$ is a general notation for a polynomial in the coefficients $R_1$ ... $R_m$, such that the subscript sum of each term is equal to $m$. Further, if distinction is necessary, $(R')_m$, $(R'')_m$ may denote similar polynomials, non-identical with $(R)_m$. Later we shall denote with $(R)'_m$ such polynomial, where the coefficients $R_1$ ... $R_{m-1}$ appear only, and with $(R)^{\cdot}_m$ such polynomial in $R_1$ ... $R_m$ and at least one of its derivatives $\dot R_1$ ... $\dot R_m$.

We may also obtain directly from (6)

$$\begin{aligned} \frac{d\dot U}{d\varrho} + \frac{dW}{d\varrho}\frac{d\dot W}{d\varrho} = \frac{S^n}{2}\{1 + (2n-3)R_1\varrho^2 + [(3n-6)R_2 + (4n^2 - 13n + 9)R_1^2/2]\varrho^4 + \\ \cdots + (R')_m\varrho^{2m} + \cdots\} \end{aligned} \tag{11}$$

Identification of the coefficients of (10) and (11) yields

$$\frac{1}{2S^{n+1}}\left(\frac{1}{S}\right)^{\cdot}+3(2n+3)R_1=(2n-3)R_1$$

$$-\frac{1}{2S^{n+1}}\left[2R_1\left(\frac{1}{S}\right)^{\cdot}+\frac{\dot R_1}{S}\right]+5\left[(3n+6)R_2+(4n^2+11n-15)R_1^2/2\right]=$$
$$=(3n-6)R_2+(4n^2-13n+9)R_1^2/2 \tag{12}$$

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$$-\frac{1}{2S^{n+1}}\left[(R)_{m-1}\left(\frac{1}{S}\right)^{\cdot}+(\dot R)_{m-1}\frac{1}{S}\right]+(R')_m=(R'')_m$$

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The equations (12) may be solved for $R_1$, $R_2$ ... taking account of the fact that

$$(R')_m-(R'')_m=2m(m+1)(n+3)R_m+(R)'_m$$

Thus we obtain

$$R_1=-\frac{1}{8(n+3)S^{n+1}}\left(\frac{1}{S}\right)^{\cdot}=\frac{\dot S}{8(n+3)S^{n+3}}$$

$$R_2=\frac{1}{12(n+3)}\left\{\frac{1}{2S^{n+1}}\left[2R_1\left(\frac{1}{S}\right)^{\cdot}+\frac{\dot R_1}{S}\right]-2(4n^2+17n-21)R_1^2\right\}$$
$$=\frac{\ddot S}{192(n+3)^2S^{2n+5}}-\frac{(2n^2+11n+3)\dot S^2}{128(n+3)^3S^{2n+6}} \tag{13}$$

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$$R_m=\frac{1}{2m(m+1)(n+3)}\left\{\frac{1}{2S^{n+1}}\left[(R)_{m-1}\left(\frac{1}{S}\right)^{\cdot}+(\dot R)_{m-1}\frac{1}{S}\right]-(R)'_m\right\}$$

The system (13) is recurrent for the coefficients $R_m$ so that $R_m$ is expressible in terms of $S$ and its derivatives up to the $m$:th order. Moreover, reinserting (13) in (8) and (7), we may satisfy the boundary condition $\dot U=0$ for $\varrho=\varrho_1$ and hence arrive at the relation

$$1+\frac{(2n+3)\dot S\varrho_1^2}{8(n+3)S^{n+3}}+\left[\frac{(3n+6)\ddot S}{192(n+3)^2S^{2n+5}}-\frac{(2n^3+22n^2+57n+63)\dot S^2}{128(n+3)^3S^{2n+6}}\right]\varrho_1^4+\cdots \tag{14}$$

a differential equation of infinite order for the determination of $S$.

Now, however, from (13) it is obvious that if $R_1=$ constant, independent of $t$, then $R_2$, $R_3$ ... $R_m$ will also be constants. Let us put $R_1=-C/\varrho_1^2$, where $C$ is a certain constant, then we have from (13)

$$\frac{\dot S\varrho_1^2}{8(n+3)S^{n+3}}=-C \tag{15}$$

and from (14) the relation

$$1-(2n+3)C-\frac{3(2n^2+5n+9)}{2(n+3),}C^2\ldots=0 \tag{16}$$

This power series in $C$ will converge for any constant $n$. It may then be solved for $C$, viz. for the smallest root of (16)

$$C=\frac{1}{2n+3}-\frac{3(2n^2+5n+9)}{2(n+3)(2n+3)^3}-\ldots \tag{17}$$

From this formula we obtain the $C$-values for $n \geqslant 5$ in the following table

| | | | | | | |
|---|---|---|---|---|---|---|
| $n=$ | 1 | 3 | 5 | 7 | 10 | $\infty$ |
| $C=$ | 0.16 | 0.096 | 0.0695 | 0.0540 | 0.0410 | 0 |
| $8(n+3)C=$ | 5.1 | 4.6 | 4.45 | 4.36 | 4.27 | 4 |
| $8(n+3)(n+2)C=$ | 15 | 23 | 31.2 | 39.2 | 51.2 | $\infty$ |

The convergence of equation (16) is slow and the accuracy of (17) is poor, particularly for the smaller values of $n$.

In the case $n=1$ it is easy to obtain a more accurate formula. This case has less interest from the physical point of view. For most engineering materials and for structural metals in particular, we have $n=2\cdots10$, [2]. In the case $n=1$ it is easy to obtain some more terms in equation (13), say, the expressions for $R_3$, $R_4$ and $R_5$. We thus obtain, instead of (16)

$$C=\frac{1}{5}-\frac{6}{5}C^2-\frac{169}{90}C^3-\frac{289}{90}C^4-\frac{518}{45}C^5\ldots \tag{18}$$

This equation gives $C=0.159$ or rather 0.16 with two correct figures instead of $C=0.15$, using equation (17).

In the case $n=3$ we may obtain similarly

$$C=\frac{1}{9}-\frac{7}{6}C^2-\frac{115}{36}C^3-\frac{1023}{90}C^4\ldots \tag{19}$$

from which we may compute the value $C=0.096$ instead of 0.097 from equation (17).

It turns out that the product $8(n+3)C$ varies but slightly with $n$ from $n=1$ to $\infty$. An average value of 4.5 would suit most structural metals in the technically important temperature range with an accuracy of about 5 per cent.

Now equation (15) may be integrated, taking account of the initial condition $S=S_0$ for $t=0$. Thus we obtain

$$S=[1/S_0^{n+2}+8(n+2)(n+3)Ct/\varrho_1^2]^{-1/(n+2)} \tag{20}$$

From (13) and (15) follows

$$R_1=-C/\varrho_1^2$$

$$R_2=\frac{(4n^2+21n-9)C^2}{6(n+3)\varrho_1^4} \tag{21}$$

$$\cdots\cdots\cdots\cdots$$

$$R_m = \frac{2\,(R)_{m-1}\,C}{m\,(m+1)\,\varrho_1^2}$$

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and so

$$R = S\left\{1 - C\left(\frac{\varrho}{\varrho_1}\right)^2 + \frac{(4n^2+21n-9)\,C^2}{6\,(n+3)}\left(\frac{\varrho}{\varrho_1}\right)^4 + \cdots\right\}$$
$$T = S\left\{1 - 3C\left(\frac{\varrho}{\varrho_1}\right)^2 + \frac{5\,(4n^2+21n-9)\,C^2}{6\,(n+3)}\left(\frac{\varrho}{\varrho_1}\right)^4 + \cdots\right\} \qquad (22)$$

We see that for $t=0$, the dimensionless stresses $R$ and $T$ deviate appreciably from the common value $S_0$ only in the outer parts of the membrane. The form of the membrane may be obtained from (5) and (22) by integration, satisfying the boundary condition $W=0$ for $\varrho=\varrho_1$, viz.

$$W = \frac{\varrho_1^2}{2S}\left\{\frac{1}{2}\left[1-\left(\frac{\varrho}{\varrho_1}\right)^2\right] + \frac{C}{4}\left[1-\left(\frac{\varrho}{\varrho_1}\right)^4\right] - \frac{4n^2+15n-27}{36\,(n+3)}\,C^2\left[1-\left(\frac{\varrho}{\varrho_1}\right)^6\right] - \cdots\right\} \qquad (23)$$

This gives the character of the solution found. For $t=0$, we have a state of stress according (22) and a deflected form of the middle surface according to (23), assuming $S=S_0$. The greater $S_0$, the smaller $W$ for $t=0$. With increasing $t$, the quantity $S$ will decrease from $S=S_0$ to zero for $t=\infty$, while the stresses drop and the deflection increases. The form of the middle surface is very nearly parabolic throughout the whole process.

An originally plane membrane is not included in our solution, as it would correspond to $S_0=\infty$. In order to comply with an initial condition $W=0$ for $t=0$, it would be necessery to take account of creep in bending. This action will be discussed in future work.

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## REFERENCES

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